ALLOWING FOR THE INJECTION OF GAS WHEN CALCULATING SUPERSONIC FLOW AROUND A CONE

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Supersonic flow around a cone is analyzed with due allowance for injection affects. Equations are obtained for the pressure at the contact surface and also the position of the condensation jump (shock wave) and its departure from the vertex of the cone. Some numerical results of the calculations are presented.

A theoretical solution to the problem of intensive uniform injection through the walls of a porous cone of infinite length was given in [1, 2] on the assumption of a detached boundary layer and supersonic flow around the latter. The solution was reduced to a determination of the position of the contact surface separating the flow of injected gas from the outer conical flow. The a priori assumption [1, 2] as to the rectilinear nature of the separating surface of the current agrees with experiment [3, 4], and for subsonic injection normal to the surface of the cone determines its angle of inclination in relation to the aperture angle of the cone in the form

$$\ln\frac{1+\cos(\delta+\Delta\theta)}{1-\cos(\delta+\Delta\theta)}+\frac{2\cos(\delta+\Delta\theta)}{1-\cos^2(\delta+\Delta\theta)}=\ln\frac{1+\cos\delta}{1-\cos\delta}-\frac{2}{\cos\delta}. \tag{1}$$

Subsequently, on the basis of his own experiments [4] and those of Otis [5], Bott showed that the position of the contact surface depended, not only on the aperture angle of the cone, but also on the mass flow and molecular weight of the injected gas, the temperature of the wall of the cone, and the pressure at the contact surface. The latter parameter in turn depends on the conditions of the incident flow, the aperture angle of the condensation jump, and the effects of viscous interaction.

The present article constitutes a development of the earlier papers. We shall show how to determine the pressure at the contact surface, the aperture angle of the shock wave, and its departure from the vertex of the cone on the basis of certain experimental constants relating to the injection of gas into a supersonic flow and the results of the constant-density theory [6, 7].

Let us consider an approximate solution to the problem of intense normal injection into a supersonic flow at the wall of the cone, starting from semiempirical data based on the theorem of the momentum increment, as applied to an arbitrary volume OBC of the internal flow (Fig. 1). In this analysis the control surface of the volume should be formed by the wall of the cone OB, the contact surface OC, and a surface obtained by rotating the element BC perpendicular to the wall around the axis of the cone. For uniform injection and a rectilinear separating surface of the current, the gas-dynamic parameters p_δ, ρ_δ, v_δ, T_δ and $p_{\delta+\Delta\theta}$ are independent of r and may be taken as constant. The pressure, density, and velocity of flow in the section BC vary with the polar angle θ. However, if we consider simply the average values of these and regard the flow in the range δ-$(\delta+\Delta\theta)$ as subsonic and adiabatic, we may easily derive the following from the equation of continuity and the equation of the change in momentum in the volume OBC in the direction of the cone wall

$$\frac{m^2}{p^2_{\delta+\Delta\theta}}\cdot\frac{RT_\delta}{\mu\left(2\operatorname{tg}\Delta\theta+\frac{\operatorname{tg}^2\Delta\theta}{\operatorname{tg}\delta}\right)^2}=\varphi(1-\varphi). \tag{2}$$

Changes in φ may cause changes in $\varphi(1-\varphi)$. Experimental results [5] relating to the injection of gas into a supersonic flow for different Reynolds numbers per unit length of the cone, expressed in coordinates

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of $(2\mathrm{tg}\Delta\theta + \mathrm{tg}^2\Delta\theta/\mathrm{tg}\,\delta)$ and $m/p_{\delta+\Delta\theta}(RT\delta/\mu)^{1/2}$, show that $\varphi(1-\varphi)$ remains constant over a wide range of variation of the flows of injected gas.

According to experiment, the function $\varphi(1-\varphi) = 0.152$ and almost corresponds to its mean value of $\int\limits_0^1 \varphi(1-\varphi)d\varphi = 0.167$ as φ varies from 0 to 1.

Let us introduce the notation

$$\eta = \frac{p_{\delta+\Delta\theta}}{m}\left(\frac{\mu C}{RT_\delta}\right)^{\frac{1}{2}}, \quad C = \varphi(1-\varphi).$$

From Eq. (2) we find

$$\mathrm{tg}\,\Delta\theta = -\mathrm{tg}\,\delta \pm \left[\mathrm{tg}\,\delta\left(\mathrm{tg}\,\delta + \frac{1}{\eta}\right)\right]^{\frac{1}{2}}. \tag{3}$$

Since $\eta > 0$, we reject the second root as physically unrealistic, i.e., we have finally

$$\mathrm{tg}\,\Delta\theta = \mathrm{tg}\,\delta\left[\left(1 + \frac{1}{\eta\,\mathrm{tg}\,\delta}\right)^{\frac{1}{2}} - 1\right]. \tag{4}$$

An increase or decrease in the angle $\Delta\theta$ of inclination of the contact surface to the surface of the cone for fixed angles δ may be explained as being due to a decrease or increase in the parameter η. For large η the flow model is such that the effects of mass transfer appear in the boundary layer. For η of the order of 0.5-1.0 the contact surface breaks up [4, 5], and this corresponds to the onset of large-scale mixing between the injected gas and the gas of the external conical flow.

For these limiting cases, on the assumption of a thin cone, we easily derive the following from Eq. (4):

$$\Delta\theta \cong C_1 T_\delta^{1/2} m p_{\delta+\Delta\theta}^{-1},$$

$$\Delta\theta \cong C_2 T_\delta^{\frac{1}{4}} m^{\frac{1}{2}} \delta^{\frac{1}{2}} p_{\delta+\Delta\theta}^{-\frac{1}{2}} - \delta. \tag{6}$$

Here $C_1 = (R/4\mu C)^{1/2}$; $C_2 = (R/\mu C)^{1/4}$ are constants for any specified injected gas.

In the problem of intensive injection, a certain difficulty arises in determining the pressure $p_{\delta+\Delta\theta}$, since this is related to the parameter η. Let us find the value of this for an external flow corresponding to hypersonic velocities. According to the assumption that a constant density exists behind the shock wave and that a $\varepsilon \ll 1$, we have [6]

$$p_{\delta+\Delta\theta} = \gamma M_\infty^2 p_\infty \left(1 - \frac{3}{4}\varepsilon\right)\sin^2\beta + p_\infty, \tag{7}$$

where

$$\varepsilon = \frac{\gamma-1}{\gamma+1}\left[1 + \frac{2}{(\gamma-1)M_\infty^2\sin^2\beta}\right].$$

The pressure at the contact surface (without allowing for viscous interaction) depends on the pressure and Mach number of the unperturbed flow and the position of the shock wave, which is in turn associated with the injection parameters.

In [7] the pressure coefficient behind a conical shock wave was shown to be independent of the Mach number of the incident flow. In relation to our present problem the results of [7] may be written:

$$\sin^2\beta = (\delta + \Delta\theta)^{1.87} + M_\infty^{-2}. \tag{8}$$

Here δ and $\Delta\theta$ are taken in radians. The increment to the angle $\Delta\theta$ associated with injection may be determined from Eq. (4), or in limiting cases from Eqs. (5) and (6). For injection in which the flow is described by Eqs. (5) and (6), the latter result may be written in the form

$$\sin^2\beta = (\delta + C_1 T_\delta^{1/2} m p_{\delta+\Delta\theta}^{-1})^{1.87} + M_\infty^{-2}, \tag{9}$$

$$\sin^2\beta = \left(C_2 T_\delta^{\frac{1}{4}} m^{\frac{1}{2}} \delta^{\frac{1}{2}} p_{\delta+\Delta\theta}^{-\frac{1}{2}}\right)^{1,87} + M_\infty^{-2}. \tag{10}$$

Clearly for large η the value of $C_1 T_\delta^{1/2} m (p_{\delta+\Delta\theta}\delta)^{-1} \ll 1$, and to a first approximation it is sufficient that Eq. (9) should be satisfied in the form

$$\sin^2\beta = \delta^{1,87} + 1{,}87 C_1 T_\delta^{1/2} m \delta^{0,87} p_{\delta+\Delta\theta}^{-1} + M_\infty^{-2}. \tag{11}$$

The following condition follows directly from (7) and (11)

$$\frac{p_{\delta+\Delta\theta}}{p_\infty} = \frac{M_\infty^2 \delta^{1,87}\gamma(\gamma+7)+\gamma^2+5\gamma+4}{8(\gamma+1)}\left[1+\left(1+\frac{30\gamma(\gamma+7)(\gamma+1)M_\infty^2 C_1 T_\delta^{1/2} m\delta^{0,87}}{p_\infty\left[M_\infty^2\delta^{1,87}\gamma(\gamma+7)+\gamma^2+5\gamma+4\right]^2}\right)^{\frac{1}{2}}\right]. \tag{12}$$

This expression agrees with the theory of constant density. For the value of $\eta \to \infty$ the expression takes the asymptotic form

$$\frac{p_\delta}{p_\infty} = \frac{M_\infty^2\delta^{1,87}\gamma(\gamma+7)+\gamma^2+5\gamma+4}{4(\gamma+1)}. \tag{13}$$

On comparing the calculated data (without injection) based on Eq. (13) and the exact values given in [8], we find good agreement for $M_\infty > 3$. The close agreement thus obtained between these results, even for fairly small values of M_∞, explains the agreement between Eq. (12) and experimental data for any values of $\eta\delta > 1.1$.

For small values of $\eta\delta$ we obtain the following from (7) and (10) after some simple transformations

$$\frac{p_{\delta+\Delta\theta}}{p_\infty} = \frac{\gamma(\gamma+7)M_\infty^2\left(C_2 m^{\frac{1}{2}}\delta^{\frac{1}{2}}T_\delta^{\frac{1}{4}}\right)^{1,87}}{4(\gamma+1)p_{\delta+\Delta\theta}^{0,935}} + \frac{\gamma^2+5\gamma+4}{4(\gamma+1)}. \tag{14}$$

For known p_∞, M_∞, C_2, m, δ, T_δ the pressure at the contact surface may be determined from this relationship by trial and error.

It is easy to show that, for the case $M_\infty \gg 1$ and $\eta \cong 1$, the first term on the right-hand side of Eq. (14) is of the order of $M_\infty^2\delta$. Hence for $M_\infty^2\delta \gg 1$ we obtain the following approximate result from (14):

$$p_{\delta+\Delta\theta} \cong \left[\frac{\gamma(\gamma+7)}{4(\gamma+1)}\, p_\infty M_\infty^2 (\delta m)^{0,935} C_2^{1,87} T^{0,467}\right]^{0,517}. \tag{15}$$

Equations (12) and (14), together with Eqs. (4) and (8), determine the aperture angle of the shock wave when gas is injected into the conical axially-symmetrical flow.

We may also demonstrate the effect of the parameter η on the departure of the jump (shock wave) from the vertex of the cone, applying the approximations of the theory of constant density [6] to the external conical flow. Then the maximum value of the angle of the cone with the associated jump for subsonic injection is determined by the equation

$$\operatorname{tg}(\delta+\Delta\theta)_m = \frac{\sqrt{2}(2-\varepsilon)}{4\sqrt{\varepsilon}}. \tag{16}$$

The departure of the shock wave from the nose of the cone given by Eq. (16) for the corresponding δ_m is strictly valid under hypersonic conditions.

The combination of Eqs. (8) and (16) and the equation for ε yields the result:

$$\operatorname{tg}(\delta+\Delta\theta)_m = \frac{1}{2}\left\{1-\frac{\gamma-1}{2(\gamma+1)}-\frac{1}{(\gamma+1)\left[M_\infty^2(\delta+\Delta\theta)^{1,87}+1\right]}\right\}\left\{\frac{\gamma-1}{2(\gamma+1)}+\frac{1}{(\gamma+1)\left[M_\infty^2(\delta+\Delta\theta)^{1,87}+1\right]}\right\}. \tag{17}$$

The angles δ_m may be found for various numbers M_∞ and parameters η by trial and error from Eqs. (17) and (4). The problem of finding δ_m is simplified if the initial angular increment $\Delta\theta$ in Eq. (17) is specified for the limiting cases of injection, i.e.,

$$\operatorname{tg}\Delta\theta \cong \frac{1}{2\eta} \text{ and } \operatorname{tg}\Delta\theta \cong \left(\frac{\operatorname{tg}\delta}{\eta}\right)^{\frac{1}{2}} - \operatorname{tg}\delta$$

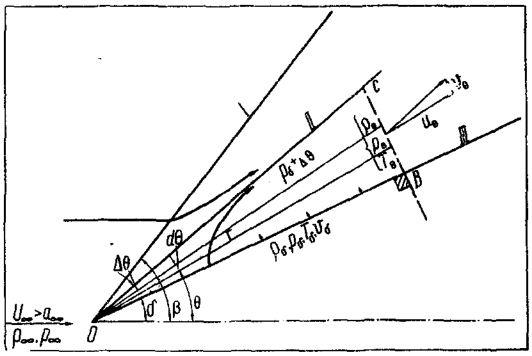

Fig. 1

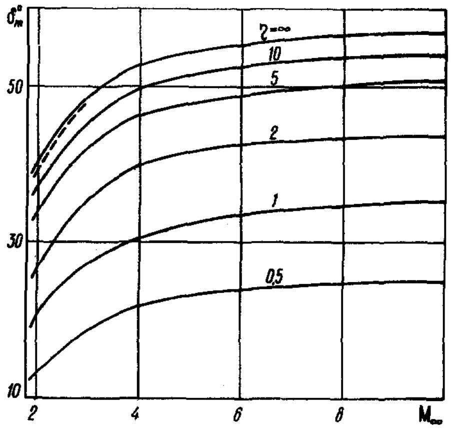

Fig. 2

Fig. 1. Flow pattern around a cone due to a supersonic flow of gas and injection normal to the surface of the wall: I) shock wave; II) surface of separation; III) wall of porous cone.

Fig. 2. Semiaperture angle of the cone δ_m corresponding to the position of the departure of the jump (shock wave) from the vertex of the cone in relation to M_∞ and η.

the argument respectively taking the values

$$\eta \operatorname{tg} \delta > 1 \text{ and } \eta \operatorname{tg} \delta < 1.$$

The results of some calculations based on Eqs. (17) and (4) are presented in Fig. 2 in the form of a dependence of δ_m on the Mach number M_∞ of the unperturbed flow for six values of η.

The broken line represents calculations corresponding to the exact solution of the case in which a supersonic flow occurs around a cone without any injection.

We see from Fig. 2 that, although the results for $\eta = \infty$ were obtained under hypersonic conditions, good agreement with the exact solution (broken line) is achieved even for fairly small supersonic M_∞ numbers. This agreement should clearly also apply to the case of $\eta < \infty$.

Finally we should point out that our analysis of the theoretical solution of [1, 2] corresponds to injection with $\eta\delta \ll 1$ and $\eta \cong 0.1$-0.3. This reduces the approximate relation (4) to Eq. (1) for

$$\eta = \frac{\Phi\delta}{\sqrt{\Phi(1-\Phi^2)^{1/2} - \delta(1-\Phi^2)}}, \tag{18}$$

where

$$\Phi = \frac{A}{4} - \frac{(16+A^2)^{\frac{1}{2}}}{2} \cos\left[60^\circ + \frac{1}{3} \arccos \frac{A^3}{(16+A^2)^{3/2}}\right],$$

$$A = \ln \frac{1+\cos\delta}{1-\cos\delta} - \frac{2}{\cos\delta}.$$

The latter equation (18) may be obtained on the basis of [9] when Eq. (1) is expressed in series in powers of $\cos(\delta + \Delta\theta)$, neglecting terms in the expansion of $\cos(\delta + \Delta\theta)$ of the fifth and higher orders.

NOTATION

U_∞ is the velocity of the unperturbed flow;
β is the semiaperture angle between the directions of the jump (shock wave) and the unperturbed flow;
δ is the semiaperture angle of the cone;
$(\delta + \Delta\theta)$ is the semiaperture angle of the surface separating the layer of injected gas from the external flow;
v_θ, u_θ are the velocity components in the directions θ and r;

r, θ	are the polar coordinates;
ρ, p, T	are the density, pressure, and temperature;
μ	is the molecular weight of the injected gas;
$R = 8.314 \cdot 10^3$ J/deg K · mole	is the gas constant;
$\gamma = C_p/C_v$;	
$\varphi = \bar{p}_\theta/p_{\delta+\Delta\theta}$;	
$\bar{p}_\theta = (1/\Delta\theta)\int_\delta^{\delta+\Delta\theta} p\,d\theta$	is the average pressure in the section BC (Fig. 1);
$m = \rho_\delta v_\delta$;	
$\varepsilon = \rho_\infty/\rho_\beta$	is the ratio of the densities in front of and behind the shock wave.

Subscripts

δ, θ, $\delta + \Delta\theta$, ∞	respectively represent conditions on the cone, on the ray θ, on the surface of separation, and in the unperturbed flow;
m	corresponds to the condition in which the jump (shock wave) deviates from the noise of the cone.

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